

# Variable-to-Fixed Length Homophonic Coding with a Modified Shannon-Fano-Elias Code

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**Abstract**—Homophonic coding is a framework to reversibly convert a message into a sequence with some target distribution. This is a promising tool to generate a codeword with a biased code-symbol distribution, which is required for capacity achieving communication by asymmetric channels. It is known that asymptotically optimal homophonic coding can be realized by a Fixed-to-Variable (FV) length code using an interval algorithm similar to a random number generator. However, FV codes are not preferable as a component of channel codes since a decoding error propagates to all subsequent codewords. As a solution for this problem an asymptotically optimal Variable-to-Fixed (VF) length homophonic code, dual Shannon-Fano-Elias-Gray (dual SFEG) code, is proposed in this paper. This code can be interpreted as a dual of a modified Shannon-Fano-Elias (SFE) code based on Gray code. It is also shown as a by-product that the modified SFE code, named SFEG code, achieves a better coding rate than the original SFE code in lossless source coding.

## I. INTRODUCTION

In the communication through asymmetric channels, it is necessary to use codewords with a biased code-symbol distribution maximizing the mutual information between the input and the output to achieve the capacity. It is well known that biased codewords can be generated from an auxiliary code over an extended alphabet based on Gallager's nonlinear mapping [1, p.208], but its complexity becomes very large when the target distribution is not expressed in a simple rational number.

A promising solution to this problem is to use a dual of lossless coding where the encoding and the decoding are inverted. Since a lossless code converts a biased sequence into an almost uniform compressed sequence, it is natural to expect that a decoder of a lossless code can be used to generate a biased codeword. This framework is first considered in the literature of LDPC codes [2][3] and a similar idea is also proposed in polar codes [4]. In these schemes fixed-length lossless (LDPC or polar) codes are used to generate a biased codeword. A similar scheme based on a fixed-length random number generator is also found in [5].

### A. Arithmetic Coding as a Biased-codeword Generator

Whereas such a fixed-length lossless code (or a fixed-length random number generator) is convenient for theoretical analyses, it is well known that variable-length lossless code such as an arithmetic code practically achieves the almost

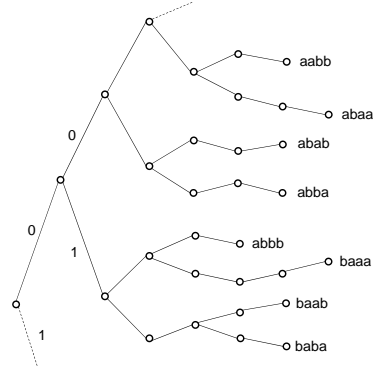


Fig. 1. A code tree of Shannon-Fano-Elias code.

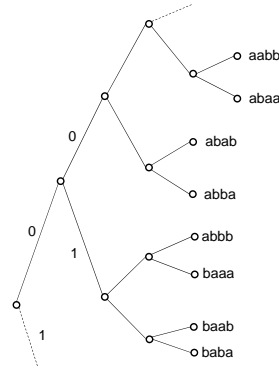


Fig. 2. A code tree of Shannon-Fano-Elias code without redundant edges.

optimal performance. Thus, it is natural to replace such fixed-length lossless codes with variable-length ones. In fact, coding schemes based on LDPC codes [6][7] and polar codes [8] where fixed-length lossless coding is replaced with arithmetic coding have been proposed in the context of lossy source coding, which can be regarded as a dual of channel coding.

Nevertheless, a naive use of an arithmetic decoder cannot be used as a generator of a biased codeword in channel coding. To see this, let us consider the code tree in Fig. 1 of Shannon-Fano-Elias code (SFE code, see, e.g. [9, Sect. 5.9] for detail) for  $(X_1, \dots, X_4) \in \{a, b\}^4$  i.i.d. from  $(P_X(a), P_X(b)) = (1/3, 2/3)$ . As we can see from the figure, the code tree of an arithmetic code is generally not complete and cannot be used as a generator of a sequence over  $\{a, b\}$  from a binary sequence. For example, the decoder of this code gets

“confused” if it receives sequence 0111 since such a sequence never appears as (a prefix of) a codeword of this code. It is theoretically possible to consider a modified SFE code where the redundant edges are removed as in Fig. 2. However it is very difficult to realize such a code with a linear complexity because the code tree is practically not stored in the memory and it is necessary to compute many other codewords to find redundant edges.

Furthermore, even if a modified SFE code without redundant edges is realized, such a code is still not appropriate as a generator of a biased sequence. For example in the code tree of Fig. 2, sequence 0101 is converted into *baaa* and therefore *baaa* appears with probability  $1/16$  which is roughly 2.5 times larger than the target probability  $P(\textit{baaa}) = 2/81$ . Such a problem generally occurs because there sometimes exists a large gap between target probabilities between adjacent sequences under the lexicographic order as in  $P_{X^4}(\textit{abbb})$  and  $P_{X^4}(\textit{baaa})$ . However, sorting of sequences (as in Shannon-Fano code) is not practical since the linear-time computability of the cumulative distribution is lost.

### B. Homophonic Codes for Channel Coding

Homophonic coding is another candidate for a generator of biased codewords. This is a framework to convert a sequence with distribution  $P_U$  reversibly into another sequence with the target distribution  $P_X$ . In particular, a homophonic code is called *perfect* if the generated sequence exactly follows the target distribution  $P_X$ .

Hoshi and Han [10] proposed a Fixed-to-Variable (FV) length perfect homophonic coding scheme based on an interval algorithm similar to a random number generator [11]. This code is applied to generation of biased codewords in [12][13] but these FV channel codes suffer the following problem for practical use. When we use an FV homophonic code to generate a biased codeword with block length  $n$ , an  $m$ -bit message is sometimes converted into one block of codeword and is converted into two blocks another time. Thus, if a decoding error occurred in one block then the receiver can no more know where the codeword is separated for each  $m$ -bit message and the decoding error propagates to all the subsequent sequences. Based on this observation it is desirable to use a Variable-to-Fixed (VF) length homophonic coding scheme for a component of a channel code.

### C. VF Homophonic Coding by Gray Code

Although it is difficult to realize a perfect VF homophonic code, we can relax the problem when we consider application to channel coding. Since homophonic coding is first considered in the context of cryptography [14], the target distribution  $P_X$  is usually uniform and there is a special meaning to be perfect, that is, the output is exactly random. On the other hand in application to channel coding, the output codeword does not have to perfectly follow the target distribution and it suffices to assure that a bad codeword does not appear too frequently.

Keeping this difference of application in mind, we propose a new VF homophonic code, dual SFEG code. This code

corresponds to a dual of a modified SFE code based on Gray code [15][16], which we call SFEG code. In SFEG code, the cumulative distribution is defined according to the order induced by Gray code instead of the lexicographic order. Under this order we can assure linear-time computability of the cumulative distribution and a small gap of probabilities between adjacent sequences. Based on this property we prove that the dual SFEG code is asymptotically perfect and its coding rate is also asymptotically optimal. We also prove as a by-product that SFEG code for lossless compression achieves a better coding rate than the original SFE code.

## II. PRELIMINARIES

We use superscript  $n$  to denote an  $n$ -bit sequence such as  $x^n = (x_1, x_2, \dots, x_n) \in \mathcal{X}^n$  for some alphabet  $\mathcal{X}$ . A subsequence is denoted by  $x_i^j = (x_i, x_{i+1}, \dots, x_j)$  for  $i \leq j$ . Let  $X^n = (X_1, X_2, \dots, X_n) \in \mathcal{X}^n$  be a discrete i.i.d. sequence, whose probability mass function is denoted by  $p(x) = \Pr[X_i = x]$ ,  $i = 1, 2, \dots, n$ , and  $p(x^n) = \Pr[X^n = x^n] = \prod_{i=1}^n p(x_i)$ . For notational simplicity we consider the case  $\mathcal{X} = \{0, 1\}$  and assume  $p(0), p(1) \in (0, 1)$  and  $p(0) \neq p(1)$ . The maximum relative gap between probability masses is denoted by  $\rho = \max\{p(0)/p(1), p(1)/p(0)\} \in (1, \infty)$ . The cumulative distribution function is defined as

$$F(x^n) = \sum_{a^n \prec x^n} p(a^n)$$

under some total order<sup>1</sup>  $\prec$  over  $\{0, 1\}^n$ . We write  $x^n + 1$  for the next sequence to  $x^n$ , that is, the smallest sequence  $y^n$  such that  $x^n \prec y^n$ . Sequence  $x^n - 1$  is defined in the same way.

### A. Shannon-Fano-Elias Code

Shannon-Fano-Elias (SFE) code is a lossless code which encodes input  $x_{(i)}^n$ ,  $i = 1, 2, \dots$ , into  $\phi_{\text{SFE}}(x_{(i)}^n) = \lfloor F(x_{(i)}^n) - 1 \rfloor + p(x_{(i)}^n)/2 \rfloor_{\lceil -\log p(x_{(i)}^n) \rceil + 1}$ , where  $\lfloor r \rfloor_l$  for  $l \in \mathbb{N}$  is the first  $l$  bits of the binary expansion of  $r \in [0, 1)$ . When we define the cumulative distribution function  $F$  for the lexicographic order this code can be encoded and decoded with a complexity linear in  $n$ . The expected code length satisfies

$$\begin{aligned} \mathbb{E}[|\phi_{\text{SFE}}(X^n)|] &= \sum_{x^n} p(x^n) (\lceil \log p(x^n) \rceil + 1) \\ &< nH(X) + 2, \end{aligned} \quad (1)$$

where  $|u|$  for  $u \in \{0, 1\}^*$  denotes the length of the sequence.

### B. Gray Code

Gray code  $g(\cdot)$  is a one-to-one map over  $n$ -bit sequences. This is the XOR operation of the input sequence and its one-bit shift to the right. For example,  $g(0110) = 0110 \oplus 0011 = 0101$  and  $g(1101) = 1101 \oplus 0110 = 1011$ , where  $\oplus$  is the bit-wise addition over  $\text{GF}(2)$ . Table I shows the output of Gray code for 3-bit sequences. The most important property of Gray code is that if  $x^n$  and  $y^n$  are adjacent in the lexicographic order then  $g(x^n)$  and  $g(y^n)$  differ only in one bit as seen from the table.

<sup>1</sup>In this paper we always write  $x \prec y$  including the case  $x = y$ .

We define *Gray order*  $\prec$  as the total order induced by Gray code, that is,  $x^n \prec y^n$  if and only if  $g^{-1}(x^n) \prec_L g^{-1}(y^n)$  where  $\prec_L$  is the lexicographic order. For example, we have  $000 \prec 001 \prec 011 \prec 010 \prec 110 \prec 111 \prec 101 \prec 100$  from Table I. Gray order is represented in a recursive way

$$x^n \prec y^n \Leftrightarrow \{x_1 \not\prec_L y_1\} \cup \{x_1 = y_1 = 0, x_2^n \prec y_2^n\} \\ \cup \{x_1 = y_1 = 1, y_2^n \prec x_2^n\}.$$

From this expression the cumulative distribution of  $X^n$  under Gray order can be computed in a linear time in  $n$  as follows.

$$\Pr[X^n \prec x^n] = \begin{cases} p(0) \Pr[X_2^n \prec x_2^n], & x_1 = 0, \\ p(0) + p(1) \Pr[X_2^n \succ x_2^n], & x_1 = 1, \end{cases} \\ \Pr[X^n \succ x^n] = \begin{cases} p(1) + p(0) \Pr[X_2^n \succ x_2^n], & x_1 = 0, \\ p(1) \Pr[X_2^n \prec x_2^n], & x_1 = 1. \end{cases}$$

In the following we always assume that  $x^n \in \{0, 1\}^n$  is aligned by Gray order and write  $F(x^n) = \Pr[X^n \prec x^n]$  for the cumulative distribution function under this order. Similarly to the computation of  $F(\cdot)$  we can show that its inverse  $F^{-1}(r) = \min\{x^n : F(x^n) > r\}$  is also computed in a linear time. From the property of Gray code we always have

$$1/\rho \leq p(x^n - 1)/p(x^n) \leq \rho. \quad (2)$$

### C. Homophonic Coding

Let  $\mathcal{U}$  and  $\mathcal{X}$  be the input and output alphabets of sequences. Let  $\mathcal{I} \subset \mathcal{U}^*$  be a set such that for any sequence  $u^\infty \in \mathcal{U}^\infty$  there exists  $m > 0$  such that  $u_1^m \in \mathcal{I}$ . A homophonic code  $\phi$  is a (possibly random) map from  $\mathcal{I} \subset \mathcal{U}^*$  onto  $\mathcal{O} \subset \mathcal{X}^*$ . A homophonic code is called *perfect* with respect to the pair of distributions  $(P_U, P_X)$  if  $\phi(U_{(1)}^{m_1})\phi(U_{(2)}^{m_2}) \cdots$  is i.i.d. from  $P_X$  for the input  $U_{(1)}^{m_1}U_{(2)}^{m_2} \cdots$  i.i.d. from  $P_U$ . We define that a homophonic code is *weakly  $\delta$ -perfect* in the sense of max-divergence if

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sup_{x^k: P_{\tilde{X}^k}(x^k) > 0} \log \frac{P_{\tilde{X}^k}(x^k)}{P_{X^k}(x^k)} \leq \delta, \quad (3)$$

where  $\tilde{X}^k$  is the first  $k$  symbols of the sequence  $\phi_n(U_{(1)}^{m_1})\phi_n(U_{(2)}^{m_2}) \cdots$ . We call this notion “weakly” perfect since a perfect homophonic code is a code satisfying the condition such that  $\delta = 0$  and  $\limsup_{k \rightarrow \infty}$  is replaced with  $\sup_{k \in \mathbb{N}}$  in (3).

A weakly perfect homophonic code can be used as a component of a capacity-achieving channel code in the following way. Assume that there exists a VF weakly  $\delta$ -perfect homophonic code with output length  $n$  and a channel code with block length  $n'$  such that the decoding error probability is  $\epsilon_{n'}$  satisfying  $\lim_{n' \rightarrow \infty} \epsilon_{n'} = 0$  under some ideal codeword

TABLE I  
GRAY CODE.

$x^n$	000	001	010	011	100	101	110	111
$g(x^n)$	000	001	011	010	110	111	101	100

### Algorithm 1 Decoding of SFEG Code

**Input:** Received sequence  $u^\infty \in \{0, 1\}^\infty$ .

```

1:  $i := 1, j := 1$ .
2: loop
3:    $r := 0.u_j u_{j+1} \cdots$  and  $\hat{x}^n := F^{-1}(r)$ .
4:    $\hat{l} := \bar{l}(\hat{x}^n)$  and  $\hat{F} = \lfloor F(\hat{x}^n - 1) + p(\hat{x}^n)/2 \rfloor_{\hat{l}}$ .
5:   if  $r \geq \hat{F} + 2^{-\hat{l}}$  then
6:      $\tilde{x}^n := \hat{x}^n + 1$ .
7:   else if  $r < \hat{F}$  then
8:      $\tilde{x}^n := \hat{x}^n - 1$ .
9:   else
10:     $\tilde{x}^n := \hat{x}^n$ .
11:  end if
12:  Output  $\tilde{x}_{(i)}^n := \tilde{x}^n$ .
13:   $i := i + 1, j := j + \bar{l}(\tilde{x}^n)$ .
14: end loop
```

distribution. Since the decoding error probability of the  $n$ -block sequence of the channel codewords is at most  $n\epsilon_{n'}$  under the ideal distribution, the decoding error probability of the sequence generated by the VF homophonic code is roughly bounded by  $n\epsilon_{n'}2^{n\delta}$ . Thus the decoding error probability can be arbitrarily small when  $n'$  is sufficiently large with respect to  $n$ . Based on this argument we can easily construct a VF channel code achieving the capacity by, e.g., replacing the FV homophonic code used as a component for the capacity-achieving channel code in [13] with such a weakly  $\delta$ -perfect VF homophonic code. In this paper we construct a weakly  $\delta_n$ -perfect VF homophonic code such that  $\delta_n = O(1/n)$ .

### III. SHANNON-FANO-ELIAS-GRAY CODE

In this section we propose Shannon-Fano-Elias-Gray (SFEG) code as a simple modification of SFE code. This encodes  $x_{(i)}^n$  into  $\phi_{\text{SFEG}}(x_{(i)}^n) = \lfloor F(x_{(i)}^n - 1) + p(x_{(i)}^n)/2 \rfloor_{\bar{l}(x_{(i)}^n)}$ , where  $\bar{l}(x^n) = \lceil -\log \gamma p(x^n) \rceil + 1$  and  $\gamma = (1 + \rho)/\rho \in (1, 2)$ . There are only two differences from the SFE encoder: the cumulative distribution  $F$  is defined by Gray order and there is a factor  $\gamma$  in the code length  $\bar{l}(x^n)$ . The decoding of this code is given in Algorithm 1. Here by abuse of notation we sometimes identify  $\lfloor r \rfloor_l$ , the first  $l$  bits of the binary expansion of  $r$ , with the real number  $2^{-l} \lfloor 2^l r \rfloor$ .

**Theorem 1.** *SFEG code is uniquely decodable. Furthermore, the average code length satisfies*

$$\mathbb{E}[\|\phi_{\text{SFEG}}(X^n)\|] < nH(X) + 2 - \log((1 + \rho)/\rho). \quad (4)$$

From this theorem we see that the upper bound on the average code length of SFEG code improves that of SFE code in (1) by  $\log(1 + \rho)/\rho \in (0, 1)$ .

We prove this theorem by the following lemma.

**Lemma 1.** *If  $x + 2^{-l'} \geq x' + 2^{-\min\{l, l'\}}$  then  $\lfloor x \rfloor_l \geq \lfloor x' \rfloor_{l'}$ .*

*Proof.* This lemma is straightforward from

$$\lfloor x \rfloor_l \geq \lfloor x' \rfloor_{l'}$$

$$\begin{aligned}
&\Leftrightarrow \{l \geq l', x \geq x'\} \cup \{l < l', x \geq x' + 2^{-l} - 2^{-l'}\} \\
&\Leftrightarrow \{l \geq l', x + 2^{-l'} \geq x' + 2^{-l'}\} \\
&\quad \cup \{l < l', x + 2^{-l'} \geq x' + 2^{-l}\} \\
&\Leftrightarrow x + 2^{-l'} \geq x' + 2^{-\min\{l, l'\}}. \quad \square
\end{aligned}$$

*Proof of Theorem 1.* Eq. (4) holds since

$$\begin{aligned}
\mathbb{E}[\|\phi_{\text{SFEG}}(X^n)\|] &= \sum_{x^n} p(x^n)(\lceil -\log \gamma p(x^n) \rceil + 1) \\
&< \sum_{x^n} p(x^n)(-\log \gamma p(x^n) + 2) \\
&= nH(X) + 2 - \log((1 + \rho)/\rho).
\end{aligned}$$

We prove the unique decodability by showing that  $\hat{x}_{(i)}^n = x_{(i)}^n$  holds in Step 9 of Algorithm 1. Let  $x^n = x_{(i)}^n$  and  $G = \phi_{\text{SFEG}}(x^n) = \lfloor F(x^n - 1) + p(x^n)/2 \rfloor_{\bar{l}(x^n)}$ . Then  $r \in [G, G + 2^{-\bar{l}(x^n)})$  holds from the encoding algorithm. From the decoding algorithm, if  $r \in [F(x^n - 1), F(x^n))$  then  $(\hat{x}^n, \tilde{x}^n)$  given in Algorithm 1 satisfies  $\hat{x}^n = \tilde{x}^n = x^n$  and we consider the other case in the following.

First we consider the case  $r \in [G, F(x^n - 1))$ . Since  $\hat{x}^n \not\geq x^n$  in this case,  $\tilde{x}^n = x^n$  is equivalent to  $\{\hat{x}^n = x^n - 1, r \geq \hat{F} + 2^{-\hat{l}}\}$ , where  $\hat{F}$  is given in Algorithm 1. The former equality  $\hat{x}^n = x^n - 1$  holds since

$$\begin{aligned}
r &\geq G \geq F(x^n - 1) + p(x^n)/2 - 2^{\log \gamma p(x^n) - 1} \\
&= F(x^n - 2) + p(x^n - 1) + (1 - \gamma)p(x^n)/2 \\
&\geq F(x^n - 2) + p(x^n)/\rho + (1 - \gamma)p(x^n)/2 \\
&= F(x^n - 2) + p(x^n)/(2\rho) \\
&\geq F(x^n - 2).
\end{aligned}$$

We obtain the latter inequality  $r \geq \hat{F} + 2^{-\hat{l}}$  by letting  $\bar{F}(x^n) = F(x^n - 1) + p(x^n)/2$  and using Lemma 2 since

$$\begin{aligned}
r &\geq \hat{F} + 2^{-\hat{l}} \\
&\Leftrightarrow \lfloor \bar{F}(x^n) \rfloor_{\bar{l}(x^n)} \geq \lfloor \bar{F}(x^n - 1) \rfloor_{\bar{l}(x^n - 1)} + 2^{-\bar{l}(x^n - 1)} \\
&\Leftrightarrow \bar{F}(x^n) - \bar{F}(x^n - 1) \geq 2^{-\min\{\bar{l}(x^n - 1), \bar{l}(x^n)\}} \\
&\Leftrightarrow (p(x^n - 1) + p(x^n))/2 \geq 2^{-1 - \lceil -\log \gamma \max\{p(x^n - 1), p(x^n)\} \rceil} \\
&\Leftrightarrow (1 + 1/\rho) \max\{p(x^n - 1), p(x^n)\}/2 \\
&\quad \geq \gamma \max\{p(x^n - 1), p(x^n)\}/2 \\
&\Leftrightarrow 1 + 1/\rho \geq \gamma.
\end{aligned}$$

Finally we consider the remaining case  $r \in [F(x^n), G + 2^{-\bar{l}(x^n)})$ . In the same way as the former case we can show  $\{\hat{x}^n = x^n + 1, r < \hat{F}\}$ , which implies  $\tilde{x}^n = x^n$ .  $\square$

#### IV. DUAL SFEG CODE

In this section we construct a VF homophonic code based on SFEG code, which we call the dual SFEG code. The main difference from SFEG code is that we use the transformed cumulative distribution function  $F_I(x^n) = a + (b - a)F(x^n)$  for an interval  $I = [a, b] \subset [0, 1)$ . As we show later, the real number  $r$  corresponding to the message sequence at each iteration is uniformly distributed over an interval  $I$  that

#### Algorithm 2 Encoding of Dual SFEG Code

**Input:** Message  $u^\infty \in \{0, 1\}^\infty$ .

```

1:  $i := 1, j := 1, I := [0, 1)$ .
2: loop
3:    $r := 0.u_j u_{j+1} \dots, \hat{x}^n := F_I^{-1}(r)$ .
4:   if  $r \geq \bar{F}_I(\hat{x}^n)$  then
5:      $\tilde{x}^n := \hat{x}^n + 1$ .
6:   else
7:      $\tilde{x}^n := \hat{x}^n$ .
8:   end if
9:   Output  $\tilde{x}_{(i)}^n := \tilde{x}^n$ .
10:   $i := i + 1, j := j + l_I(\tilde{x}^n)$ .
11:   $I := [\min\{\bar{F}_I(\tilde{x}^n - 1), F_I(\tilde{x}^n - 1)\}_{l_I(\tilde{x}^n)},$ 
     $\langle \min\{\bar{F}_I(\tilde{x}^n), F_I(\tilde{x}^n)\} \rangle_{l_I(\tilde{x}^n)})$ .
12: end loop

```

#### Algorithm 3 Decoding of Dual SFEG Code

**Input:** Received sequences  $\tilde{x}_{(1)}^n, \tilde{x}_{(2)}^n, \dots \in \{0, 1\}^n$ .

```

1:  $i := 1, j := 1, I := [0, 1)$ .
2: loop
3:   Output  $\hat{u}_j^{j+l_I(\tilde{x}_{(i)}^n)-1} := \lfloor F_I(\tilde{x}^n - 1) \rfloor_{l_I(\tilde{x}_{(i)}^n)}$ .
4:    $I := [\min\{\bar{F}_I(\tilde{x}^n - 1), F_I(\tilde{x}^n - 1)\}_{l_I(\tilde{x}^n)},$ 
     $\langle \min\{\bar{F}_I(\tilde{x}^n), F_I(\tilde{x}^n)\} \rangle_{l_I(\tilde{x}^n)})$ .
5:    $i := i + 1, j := j + l_I(\tilde{x}_{(i)}^n)$ .
6: end loop

```

is generally different from  $[0, 1)$ . By using the transformed function  $F_I$  the output distribution becomes close to  $P_{X^n}$ . Let  $\langle r \rangle_l = 2^l(r - \lfloor r \rfloor_l) \in [0, 1)$  be the real number corresponding to the  $(l + 1, l + 2, \dots)$ -th bits of  $r \in [0, 1)$ . The encoding and decoding of the dual SFEG code, which are similar to the decoding and encoding of SFEG code, are given in Algorithms 2 and 3, respectively, where

$$\bar{F}_I(x^n) = \lfloor F_I(x^n - 1) + 2^{-l_I(x^n)} \rfloor_{l_I(x^n)}$$

for  $l_I(x^n) = \lfloor -\log \rho(b - a)p(x^n) \rfloor$  and  $I = [a, b)$ .

**Theorem 2.** Dual SFEG code for uniform input  $U^\infty$  with outputs  $\tilde{X}_{(1)}^n, \tilde{X}_{(2)}^n, \dots$  is (i) uniquely decodable, (ii) weakly  $(\frac{1}{n} \log 2\rho)$ -perfect and (iii) the average input length satisfies

$$\mathbb{E}[l_I(\tilde{X}^n)] > nH(X) - 1 - 2 \log \rho.$$

In the dual SFEG code the property (2) of Gray code is essentially used to prove the code is weak  $\delta$ -perfect for some  $\delta \in (0, \infty)$ . This fact contrasts with the relation between SFE code and SFEG code, where Gray code only contributes to improve the code length by  $(1 + \rho)/\rho$ .

Now define  $I(x^n) = [\min\{F_I(x^n - 1), F_I(x^n - 1)\}, \min\{\bar{F}_I(x^n), F_I(x^n)\}]$ . We show Lemmas 2–4 in the following to prove the theorem.

**Lemma 2.** For any  $x^n \in \{0, 1\}^n$  it holds that  $\bar{F}_I(x^n) \geq \max\{F_I(x^n - 1), \lfloor F_I(x^n) \rfloor_{l_I(x^n + 1)}\}$  and, consequently,

$$I(x^n) \subset [\lfloor F_I(x^n - 1) \rfloor_{l_I(x^n)}, \bar{F}_I(x^n)). \quad (5)$$

*Proof.*  $\overline{F}_I(x^n) \geq F_I(x^n - 1)$  is straightforward from the definition of  $\overline{F}_I(x^n)$  and it suffices to show  $\overline{F}_I(x^n) \geq \lfloor F_I(x^n) \rfloor_{l_I(x^n+1)}$ , which is equivalent to

$$\lfloor G + 2^{-l_I(x^n)} \rfloor_{l_I(x^n)} \geq \lfloor G + (b-a)p(x^n) \rfloor_{l_I(x^n+1)}, \quad (6)$$

where  $G = F_I(x^n - 1)$ . This holds from Lemma 2 since

$$\begin{aligned} (6) &\Leftrightarrow G + 2^{-l_I(x^n)} + 2^{-l_I(x^n+1)} \\ &\geq G + (b-a)p(x^n) + 2^{-\min\{l_I(x^n), l_I(x^n+1)\}} \\ &\Leftrightarrow 2^{-\max\{l_I(x^n), l_I(x^n+1)\}} \geq (b-a)p(x^n) \\ &\Leftrightarrow \rho \min\{p(x^n), p(x^n+1)\} \geq p(\hat{x}^n) \\ &\Leftrightarrow \rho p(x^n)/\rho \geq p(x^n). \quad \square \end{aligned}$$

**Lemma 3.** *At each loop of Algorithm 2,  $\tilde{x}^n = x^n$  if and only if  $r \in I(x^n)$ .*

*Proof.* Since  $\hat{x}^n = x^n$  if and only if  $r \in [F_I(x^n - 1), F_I(x^n))$ , we have  $\tilde{x}^n = x^n$  if and only if

$$\begin{aligned} &\{r \in [F_I(x^n - 2), F_I(x^n - 1)), r \geq \overline{F}_I(x^n - 1)\} \text{ or} \\ &\{r \in [F_I(x^n - 1), F_I(x^n)), r < \overline{F}_I(x^n)\}. \end{aligned} \quad (7)$$

Since  $\overline{F}_I(x^n - 1) \geq F_I(x^n - 2)$  holds from Lemma 1, (7) is equivalent to

$$\begin{aligned} &\overline{F}_I(x^n - 1) \leq r < F_I(x^n - 1) \text{ or} \\ &F_I(x^n - 1) \leq r < \min\{F_I(x^n), \overline{F}_I(x^n)\}. \end{aligned} \quad (8)$$

We can easily show that (8) is equivalent to  $r \in I(x^n)$  by considering cases  $\overline{F}_I(x^n - 1) \leq F_I(x^n - 1)$  and  $\overline{F}_I(x^n - 1) > F_I(x^n - 1)$  separately.  $\square$

**Lemma 4.** *At Step 3 in each loop of Algorithm 2,  $r$  is uniformly distributed over  $I$ .*

*Proof.* In the first loop  $r$  is uniformly distributed over  $I = [0, 1)$ . Assume that  $r$  is uniformly distributed over  $I$  at some loop. Then, given  $\tilde{x}^n$  is sent,  $r$  is uniformly distributed over  $I(\tilde{x}^n)$  from Lemma 3. Here we have from Lemma 2 that  $I(\tilde{x}^n) \subset [\lfloor F_I(\tilde{x}^n - 1) \rfloor_{l_I(\tilde{x}^n)}, \overline{F}_I(\tilde{x}^n))$ , which implies that the first  $l_I(\tilde{x}^n)$  bits of  $r = 0.u_j u_{j+1} \dots \in I(\tilde{x}^n)$  are unique. As a result,  $0.u_{j+l_I(\tilde{x}^n)} u_{j+l_I(\tilde{x}^n)+1} \dots$  is uniformly distributed over  $[\langle \min\{\overline{F}_I(\tilde{x}^n - 1), F_I(\tilde{x}^n - 1) \} \rangle_{l_I(\tilde{x}^n)}, \langle \min\{\overline{F}_I(\tilde{x}^n), F_I(\tilde{x}^n) \} \rangle_{l_I(\tilde{x}^n)})$ , which means that  $r$  is also uniformly distributed over  $I$  in the next loop.  $\square$

*Proof of Theorem 2.* (i) By the argument in the proof of Lemma 4,  $(u_j, u_{j+1}, \dots, u_{j+l_I(\tilde{x}^n_{(i)})-1})$  is given as  $\lfloor F_I(\tilde{x}^n_{(i)} - 1) \rfloor_{l_I(\tilde{x}^n_{(i)})}$ , which implies the unique decodability.

(ii) Let  $\tilde{p}_{(i)}(x^n)$  be the probability of the event  $\tilde{x}^n_{(i)} = x^n$ . Then it holds from (5) and Lemmas 3 and 4 that

$$\tilde{p}_{(i)}(x^n) \leq \frac{2^{-l_I(x^n)}}{b-a} = \frac{2^{-\lfloor -\log \rho(b-a)p(x^n) \rfloor}}{b-a} < 2\rho p(x^n).$$

Therefore, letting  $\tilde{X}^k$  be the first  $k$  bits of the output sequence  $\tilde{x}^n_{(1)}, \tilde{x}^n_{(2)}, \dots$  for the uniform input  $U^\infty$ , we have

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sup_{x^k} \log \frac{P_{\tilde{X}^k}(\tilde{X}^k)}{P_{X^k}(\tilde{X}^k)} \leq \limsup_{k \rightarrow \infty} \frac{1}{k} \log(2\rho)^{[k/n]}$$

$$= \frac{1}{n} \log 2\rho,$$

that is, the dual SFEG code is weakly  $(\frac{1}{n} \log 2\rho)$ -perfect.

(iii) Let  $\tilde{F}_{(i)}(x^n)$  be the cumulative distribution function induced by  $\tilde{p}_{(i)}(x^n)$ . Then  $F(x^n - 1) < \tilde{F}_{(i)}(x^n) \leq F(x^n)$  holds and therefore

$$\begin{aligned} E[l_I(\tilde{X}^n_{(i)})] &= \int_0^1 l_I(\tilde{F}_{(i)}^{-1}(r)) dr \\ &\geq \int_0^1 \min\{l_I(F^{-1}(r) - 1), l_I(F^{-1}(r))\} dr \\ &= \int_0^1 \lfloor -\log \rho \max\{p(F^{-1}(r) - 1), p(F^{-1}(r))\} \rfloor dr \\ &\geq \int_0^1 \lfloor -\log \rho^2 p(F^{-1}(r)) \rfloor dr \\ &= \sum_{x^n} p(x^n) \lfloor -\log \rho^2 p(x^n) \rfloor \\ &> nH(X) - 1 - 2\log \rho. \quad \square \end{aligned}$$

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